THE PRESSURE ON AN ELASTIC HALF-SPACE OF A STAMP WITH A WEDGE-SHAPED PLANFORM

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The solution of the problem of pressure on an elastic half-space of a stamp [die] with a flat wedge-shaped planform makes it possible to determine the law of distribution of pressure in the neighborhood of corner points for stamps of polygonal (for example, rectangular) shape. This solution can also be used for computing the impression of plates on an elastic foundation.

L.A. Galin [1] has proposed a method for solving this problem for the case when some additional loading is acting on the elastic half-space outside the region of contact with the stamp.

In carrying out the method suggested by Galin, it was found that the solution of the problem can also be determined when the indicated additional loading is absent. We thus obtain the following formula for the pressure under the stamp:

$$p(r,\ \theta) = \frac{E}{2(1-\nu^2)}\ \frac{r^{\gamma-1}f(\theta)}{(1+\cos\theta)\sqrt{\operatorname{tg}^2 {}^1/_2\alpha - \operatorname{tg}^2 {}^1/_2\theta}}$$

where 2α is the angle at the vertex of the wedge, $f(\theta)$ is some function, while γ is a quantity which changes between the limits 0 to 1 in dependence on angle α.

1. Let the region of contact S between the stamp and the elastic half-space $z \leqslant 0$ be included between two rays making an angle 2α with each other, and let the O_x-axis bisect this angle.

The solution of the problem can of course be reduced to the determination of a harmonic function over the entire space, except for the points of a planar cut having the form of the contact region and of a continuous function $\phi(x,\ y,\ z)$ satisfying the conditions

$$\varphi(x, y, 0) = \text{const} \quad \text{on} \quad (S) \tag{1.1}$$
$$\varphi'_z(x, y, 0) = 0 \quad \text{outside} \quad (S) \tag{1.2}$$
$$\operatorname{grad} \varphi \to 0 \quad \text{for} \quad x^2 + y^2 + z^2 \to \infty \tag{1.3}$$
$$\varphi (x, y, z) \not\equiv \text{const} \tag{1.4}$$

After function $\phi(x, y, z)$ has been found, the pressure under the stamp can be determined by the formula

$$p(x, y) = \frac{E}{2(1-\nu^2)} \varphi'_z(x,y,0) \tag{1.5}$$

where E and ν are elastic constants.

We introduce the spherical coordinates r, θ, ω by means of the formulas

$$x = r\cos\theta, \qquad y = r\sin\theta\cos\omega, \qquad z = r\sin\theta\sin\omega \tag{1.6}$$

Laplace's equation then takes the form

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\omega^2} = 0 \tag{1.7}$$

In three particular cases, when $\alpha = 0$, $\alpha = \pi/2$ and $\alpha = \pi$, i.e. when the contact region degenerates into a semi-axis, into a half-plane and into the entire plane, we can easily find the following exact solutions of the problem:

$$\begin{gathered}
\varphi = C\ln\operatorname{tg}\frac{\theta}{2} + C_1 \qquad \text{for} \quad \alpha = 0 \\
\varphi = Cr^{1/2}\sqrt{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\omega} - \cos\theta} + C_1 \qquad \text{for} \quad \alpha = \frac{\pi}{2} \\
\varphi = Cr\sin\theta\sin\omega + C_1 \qquad \text{for} \quad \alpha = \pi
\end{gathered} \tag{1.8}$$

where C, C_1 are arbitrary constants, to determine which certain auxiliary conditions are needed.

The particular cases indicated lead us to consider the possibility that the required function might have the form

$$\varphi = r^\gamma\Phi(\theta, \omega) + C_1 \tag{1.9}$$

where it should be assumed that $0 \leqslant \gamma \leqslant 1$ for condition (1.3) to hold good.

It can easily be verified that function ϕ will satisfy equation (1.7) if Φ (θ) is a solution of the next equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\omega^2} + \gamma(\gamma+1)\Phi = 0 \tag{1.10}$$

Bearing in mind boundary conditions (1.1) and (1.2), we conclude that the problem is reduced to finding on the sphere (Fig. 1) a function $\Phi\ (\theta,\ \omega)$ satisfying equation (1.10) and zero at the points of arc *ABC* ($|\theta| \leqslant \alpha$, $\omega = 0$, or π).

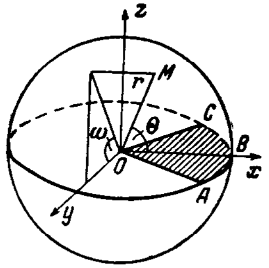

Fig. 1.

2. We make a change of variables from θ, and ω to ρ and ψ, by means of the following relations:

$$\zeta + \frac{1}{\zeta} = 2\operatorname{ctg}\frac{\alpha}{2}\operatorname{tg}\frac{\theta}{2}e^{i\omega} \qquad (\zeta = \rho e^{i\psi}) \tag{2.1}$$

By means of this transformation, the sphere with a cut along arc *ABC* is mapped into the interior of the unit circle $\rho \leqslant 1$. Equation (1.10) takes on the following form in terms of the new variables

$$L(\Phi) = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\psi^2} + \lambda f(\rho,\ \psi)\,\Phi = 0 \tag{2.2}$$

where

$$\lambda = 16\operatorname{tg}^2\frac{\alpha}{2}\,\gamma(\gamma+1) \tag{2.3}$$

$$f(\rho,\ \psi) = \frac{\rho^4 - 2\rho^2\cos 2\psi + 1}{[4\rho^2 + \operatorname{tg}^2 {}^1/_2\alpha(\rho^4 + 2\rho^2\cos^2 2\psi + 1)]^2} \tag{2.4}$$

Let function $\Phi\ (\theta,\ \omega)$ be denoted by $\Phi_1(\rho,\psi)$ after the change of variables. This function will then satisfy equation (2.2) inside the circle $\rho \leqslant 1$, and will be zero on the boundary of this circle.

Since function $\lambda\ f(\rho,\psi) \geqslant 0$, it follows that any non-zero solution of equation (2.2) which vanishes on the boundary of the region can be obtained only for a certain spectrum of values of λ.

Bearing in mind that $0 \leqslant \lambda \leqslant 1$, we obtain the next equation by means of formula (2.3):

$$\gamma = -0.5 + \sqrt{0.25 + {}^1/_{16}\lambda\operatorname{ctg}^2 {}^1/_2\alpha} \leqslant 1 \tag{2.5}$$

Out of the spectrum of values of λ indicated, those values which satisfy this last equation must be selected.

3. Function $\Phi_1(\rho, \psi)$ will satisfy the boundary condition if we assume that

$$\Phi_1(\rho, \psi) = (1 - \rho^2) F(\rho^2, \rho^2 \cos 2\psi)$$

Expanding function $F(\rho^2, \rho^2 \cos 2\psi)$ into a double Taylor series, we obtain

$$\Phi_1(\rho, \psi) = (1 - \rho^2)(a_{00} + a_{10}\rho^2 + a_{01}\rho^2 \cos 2\psi + a_{20}\rho^4 + a_{11}\rho^4 \cos 2\psi + a_{02}\rho^4 \cos^2 2\psi + \ldots) \quad (3.1)$$

We make use of the Galerkin method to determine the constants a_{00}, a_{10}, ... Let $L[\Phi_1(\rho, \psi)]$ be the result of substituting function $\Phi_1(\rho, \psi)$ into the left part of equation (2.2). If formula (3.1) stops after the first n terms, and if we require function $L[\Phi_1(\rho, \psi)]$ to be orthogonal to each of these n terms within the circle $\rho \leqslant 1$, we then obtain a system of n linear equations. Setting the determinant of the system equal to zero, we obtain an n-th degree equation in λ. Among the roots of this equation we find the one satisfying condition (2.5).

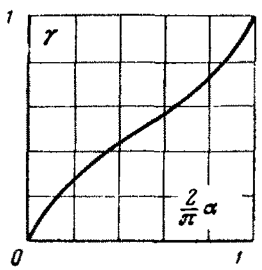

Fig. 2.

Thus, for example, in the case of a rectangular stamp ($\alpha = \pi/4$) we obtain the following six approximations for quantity γ:

$$\gamma_1 = 0.429, \qquad \gamma_2 = 0.341, \qquad \gamma_2 = 0.340,$$

$$\gamma_4 = 0.315, \qquad \gamma_5 = 0.314, \qquad \gamma_6 = 0.314,$$

When $\alpha = \pi/2$, the exact value of γ is 0.5, and an analogous sequence of approximations is the following:

$$\gamma_1 = 0.579,\ \gamma_2 = 0.517,\ \gamma_3 = 0.515,\ \gamma_4 = 0.501,\ \gamma_5 = 0.501,\ \gamma_6 = 0.500.$$

In Fig. 2 a graph is given showing the dependence of quantity γ on α. It is constructed on the basis of the following data:

$\alpha=0$	$\frac{\pi}{24}$	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$	$\frac{3\pi}{9}$	$\frac{5\pi}{12}$	$\frac{11\pi}{24}$	$\frac{\pi}{2}$
$\gamma=0$	0.13	0.23	0.31	0.38	0.44	0.50	0.56	0.62	0.69	0.77	0.87	1.0

Taking the case of a rectangular stamp with $\gamma = 0.31$, we can obtain the following formula for the pressure under the stamp

$$p(r, \theta) = \frac{E}{2(1-\nu^2)} \frac{C(1 - 0.68\,\mathrm{tg}^2\,{}^1/_2\theta - 0.4\,\mathrm{tg}^4\,{}^1/_2\theta + \ldots)}{r^{0.69}(1 + \cos\theta)\sqrt{\mathrm{tg}^2\,{}^1/_8\pi - \mathrm{tg}^2\,{}^1/_2\theta}} \qquad (3.2)$$

For comparison we give the formula obtained by Galin:

$$p(r, \theta) = \frac{E}{2(1-\nu^2)} \frac{C}{r(1 + \cos\theta)\sqrt{\mathrm{tg}^2\,{}^1/_2\alpha - \mathrm{tg}^2\,{}^1/_2\theta}} \qquad (3.3)$$

BIBLIOGRAPHY

1. Galin, L.A., *Kontaktnye zadachi uprugosti (Contact problems of elasticity)*. Moscow, 1953.

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